

VANISHING THETANULLS ON CURVES WITH INVOLUTIONS

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ABSTRACT. The configuration of theta characteristics and vanishing thetanulls on a hyperelliptic curve is completely understood. We observe in this note that analogous results hold for the σ -invariant theta characteristics on any curve C with an involution σ . As a consequence we get examples of non hyperelliptic curves with a high number of vanishing thetanulls.

1. INTRODUCTION

Let C be a smooth projective curve over \mathbb{C} . A *theta characteristic* on C is a line bundle κ such that $\kappa^2 \cong K_C$; it is even or odd according to the parity of $h^0(\kappa)$. An even theta characteristic κ with $h^0(\kappa) > 0$ is called a *vanishing thetanull*.

The terminology comes from the classical theory of theta functions. A theta characteristic κ corresponds to a symmetric theta divisor Θ_κ on the Jacobian JC , defined by a theta function θ_κ ; this function is even or odd according to the parity of κ . Thus the numbers $\theta_\kappa(0)$ are 0 for κ odd; for κ even they are classical invariants attached to the curve (“thetanullwerte” or “thetanulls”). The number $\theta_\kappa(0)$ vanishes if and only if κ is a vanishing thetanull in the above sense.

When C is hyperelliptic, the configuration of its theta characteristics and vanishing thetanulls is completely understood (see e.g. [M2]). We observe in this note that analogous results hold for the σ -invariant theta characteristics on any curve C with an involution σ . As a consequence we obtain examples of non hyperelliptic curves with a high number of vanishing thetanulls: for instance approximately one fourth of the even thetanulls vanish for a bielliptic curve.

2. σ -INVARIANT LINE BUNDLES

Throughout the paper we consider a curve C of genus g , with an involution σ . We denote by $\pi : C \rightarrow B$ the quotient map, and by $R \subset C$ the fixed locus of σ . The double covering π determines a line bundle ρ on B such that $\rho^2 = \mathcal{O}_B(\pi_* R)$; we have $\pi^* \rho = \mathcal{O}_C(R)$, $\pi_* \mathcal{O}_C \cong \mathcal{O}_B \oplus \rho^{-1}$ and $K_C = \pi^*(K_B \otimes \rho)$.

For a subset $E = \{p_1, \dots, p_k\}$ of R we will still denote by E the divisor $p_1 + \dots + p_k$. We consider the map $\varphi : \mathbb{Z}^R \rightarrow \text{Pic}(C)$ which maps $r \in R$ to the class of $\mathcal{O}_C(r)$. Its image lies in the subgroup $\text{Pic}(C)^\sigma$ of σ -invariant line bundles.

Lemma 1. φ induces a surjective homomorphism $\bar{\varphi} : (\mathbb{Z}/2)^R \rightarrow \text{Pic}(C)^\sigma / \pi^* \text{Pic}(B)$, whose kernel is $\mathbb{Z}/2 \cdot (1, \dots, 1)$.

Proof : Let K_C and K_B be the fields of rational functions of C and B respectively. Let $\langle \sigma \rangle (\cong \mathbb{Z}/2)$ be the Galois group of the covering π . Consider the exact sequence of $\langle \sigma \rangle$ -modules

$$1 \rightarrow K_C^* / \mathbb{C}^* \rightarrow \text{Div}(C) \rightarrow \text{Pic}(C) \rightarrow 0.$$

Since $H^1(\langle \sigma \rangle, K_C^*) = 0$ by Hilbert Theorem 90 and $H^2(\langle \sigma \rangle, \mathbb{C}^*) = 0$, we have $H^1(\langle \sigma \rangle, (K_C^*/\mathbb{C}^*)) = 0$, hence a diagram of exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K_B^*/\mathbb{C}^* & \longrightarrow & \text{Div}(B) & \longrightarrow & \text{Pic}(B) & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 1 & \longrightarrow & (K_C^*/\mathbb{C}^*)^\sigma & \longrightarrow & \text{Div}(C)^\sigma & \longrightarrow & \text{Pic}(C)^\sigma & \longrightarrow & 0 \end{array}$$

where the vertical arrows are induced by pull back.

If $R = \emptyset$, this shows that γ is surjective, hence there is nothing to prove. Assume $R \neq \emptyset$. Then γ is injective. Since $H^1(\langle \sigma \rangle, \mathbb{C}^*) = \mathbb{Z}/2$ and $(K_C^*)^\sigma = K_B^*$, the cokernel of α is $\mathbb{Z}/2$. The cokernel of β can be identified with $(\mathbb{Z}/2)^R$, so we get an exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow (\mathbb{Z}/2)^R \xrightarrow{\bar{\varphi}} \text{Pic}(C)^\sigma / \pi^* \text{Pic}(B) \rightarrow 0;$$

since $\mathcal{O}_C(R) \cong \pi^* \rho$, the vector $(1, \dots, 1)$ belongs to $\text{Ker } \bar{\varphi}$, and therefore generates this kernel. ■

Proposition 1. *Let M be a σ -invariant line bundle on C .*

a) *We have $M \cong \pi^* L(E)$ for some $L \in \text{Pic}(C)$ and $E \subset R$. Any pair (L', E') satisfying $M \cong \pi^* L'(E')$ is equal to (L, E) or $(L \otimes \rho^{-1}(\pi_* E), R - E)$.*

b) *There is a natural isomorphism $H^0(C, L) \cong H^0(B, L) \oplus H^0(B, L \otimes \rho^{-1}(\pi_* E))$.*

Proof : Part a) follows directly from the lemma. Let us prove b). We view $\mathcal{O}_C(E)$ as the sheaf of rational functions on C with at most simple poles along E . Then σ induces a homomorphism $\mathcal{O}_C(E) \rightarrow \sigma_* \mathcal{O}_C(E)$, hence an involution of the rank 2 vector bundle $F := \pi_* \mathcal{O}_C(E)$; thus F admits a decomposition $F = F^+ \oplus F^-$ into eigen-subbundles for this involution. The section 1 of $\mathcal{O}_C(E)$ provides a section of F^+ , which generates F^+ ; therefore $F^- \cong \det F \cong \rho^{-1}(\pi_* E)$. This gives a canonical decomposition $\pi_* \mathcal{O}_C(E) \cong \mathcal{O}_B \oplus \rho^{-1}(\pi_* E)$. Taking tensor product with L and global sections gives the required isomorphism. ■

3. σ -INVARIANT THETA CHARACTERISTICS: THE RAMIFIED CASE

In this section we assume $R \neq \emptyset$. We denote by b the genus of B and we put $r := g - 2b + 1$. By the Riemann-Hurwitz formula we have $\deg \rho = r$ and $\#R = 2r$.

We now specialize Proposition 1 to the case of theta characteristics.

Proposition 2. *Let κ be a σ -invariant theta characteristic on C .*

a) *We have $\kappa \cong \pi^* L(E)$ for some $L \in \text{Pic}(C)$ and $E \subset R$ with $L^2 \cong K_B \otimes \rho(-\pi_* E)$. If another pair (L', E') satisfies $\kappa \cong \pi^* L'(E')$, we have $(L', E') = (L, E)$ or $(L', E') = (K_B \otimes L^{-1}, R - E)$.*

b) *We have $h^0(\kappa) = h^0(L) + h^1(L)$, and the parity of κ is equal to $\deg(L) - (b - 1)$.*

Proof : a) By Proposition 1.a) κ can be written $\pi^* L(E)$, with $L \in \text{Pic}(B)$ and $E \subset R$. The condition $\kappa^2 = K_C$ translates as $\pi^*(L^2(\pi_* E)) \cong \pi^*(K_B \otimes \rho)$. Since $R \neq \emptyset$ this implies $L^2 \cong K_B \otimes \rho(-\pi_* E)$. The last assertion then follows from Proposition 1.a).

b) The value of $h^0(\kappa)$ follows from Proposition 1.b), and its parity from the Riemann-Roch theorem. ■

Lemma 2. *The group $(\text{Pic}(C)[2])^\sigma$ of σ -invariant line bundles α on C with $\alpha^2 = \mathcal{O}_C$ is a vector space of dimension $2(g-b)$ over $\mathbb{Z}/2$.*

Proof : By lemma 1 we have an exact sequence

$$(1) \quad 0 \rightarrow \text{Pic}(B) \rightarrow \text{Pic}(C)^\sigma \rightarrow (\mathbb{Z}/2)^{2r-1} \rightarrow 0.$$

For a \mathbb{Z} -module M , let $M[2] = \text{Hom}(\mathbb{Z}/2, M)$ be the kernel of the multiplication by 2 in M . Note that $\text{Ext}^1(\mathbb{Z}/2, M)$ is naturally isomorphic to $M/2M$. Applying $\text{Hom}(\mathbb{Z}/2, -)$ to (1) gives an exact sequence of $(\mathbb{Z}/2)$ -vector spaces

$$0 \rightarrow \text{Pic}(B)[2] \rightarrow (\text{Pic}(C)[2])^\sigma \rightarrow (\mathbb{Z}/2)^{2r-1} \rightarrow \text{Pic}(B)/2\text{Pic}(B) \rightarrow \text{Pic}(C)^\sigma/2\text{Pic}(C)^\sigma.$$

Let $p \in R$. The group $\text{Pic}(B)/2\text{Pic}(B)$ is generated by the class of $\mathcal{O}_B(\pi p)$; since $\pi^*(\pi p) = 2p$, this class goes to 0 in $\text{Pic}(C)^\sigma/2\text{Pic}(C)^\sigma$. Thus the dimension of $(\text{Pic}(C)[2])^\sigma$ over $\mathbb{Z}/2$ is $2b + 2r - 2 = 2(g-b)$. ■

Proposition 3. *a) The σ -invariant theta characteristics form an affine space of dimension $2(g-b)$ over $\mathbb{Z}/2$; among these, there are $2^{g-1}(2^{g-2b} + 1)$ even theta characteristics and $2^{g-1}(2^{g-2b} - 1)$ odd ones.*

b) C admits (at least) $2^{g-1}(2^{g-2b} + 1 - 2^{-r+1}\binom{2r}{r})$ vanishing thetanulls.

Proof : The σ -invariant theta characteristics form an affine space under $(\text{Pic}(C)[2])^\sigma$, which has dimension $2(g-b)$ by lemma 2.

According to Proposition 2, a theta characteristic κ is determined by a subset $E \subset R$ and a line bundle L on B such that $L^2 \cong K_B \otimes \rho(-\pi_*E)$. This condition implies $\#E \equiv r \pmod{2}$. Moreover the parity of κ is that of $\deg(L) - (b-1) = \frac{1}{2}(r - \#E)$.

Once E is fixed we have 2^{2b} choices for L . Since E and $R \setminus E$ give the same theta characteristic, we consider only the subsets E with $\#E \leq r$, counting only half of those with $\#E = r$. Thus the number of even σ -invariant theta characteristics is

$$\begin{aligned} 2^{2b} \left[\frac{1}{2} \binom{2r}{r} + \binom{2r}{r-4} + \dots \right] &= 2^{2b-3} [(1+1)^{2r} + (-1)^r(1-1)^{2r} + (-i)^r(1+i)^{2r} + i^r(1-i)^{2r}] \\ &= 2^{2b+2r-3} + 2^{2b+r-2} = 2^{g-1}(2^{g-2b} + 1), \end{aligned}$$

which gives a).

By Proposition 2. b) such a theta characteristic will be a vanishing thetanull as soon as $\deg L > b-1$, or equivalently $\#E < r$. Thus subtracting the number of theta characteristics $\kappa = \pi^*L(E)$ with $\#E = r$ we obtain b). ■

Remarks.— 1) Note that there may be more σ -invariant vanishing thetanulls, namely those of the form $\pi^*L(E)$ with $\deg L = b-1$ but $h^0(L) > 0$. These will not occur for a general (C, σ) .

2) Let $g \rightarrow \infty$ with b fixed. By the Stirling formula $\binom{2r}{r}$ is equivalent to $2^{2r}/\sqrt{\pi r}$, so $2^{-r+1}\binom{2r}{r}$ is negligible compared to $2^{g-2b} = 2^{r-1}$. Thus asymptotically we obtain $2^{2g-1-2b}$ vanishing thetanulls.

3) When $b = 0$ we recover the usual numbers for hyperelliptic curves. For $b = 1$ we obtain approximately 2^{2g-3} vanishing thetanulls, that is one fourth of the number of even theta characteristics.

4. σ -INVARIANT THETA CHARACTERISTICS: THE ÉTALE CASE

In this section we assume that σ is fixed point free ($R = \emptyset$).

Lemma 3. $(\text{Pic}(C)[2])^\sigma$ is a vector space of dimension $g + 1$ over $\mathbb{Z}/2$.

Proof : Apply $\text{Hom}(\mathbb{Z}/2, -)$ to the exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow JB \xrightarrow{\pi^*} JC^\sigma \rightarrow 0.$$

Proposition 4. a) The σ -invariant theta characteristics form an affine space of dimension $g + 1$ over $\mathbb{Z}/2$; among these, there are $3 \cdot 2^{g-1}$ even theta characteristics and 2^{g-1} odd ones.

b) C admits a set \mathcal{T} of $2^{g-2} - 2^{\frac{g-3}{2}}$ σ -invariant vanishing thetanulls; it is contained in an affine subspace of dimension $g - 1$ consisting of even theta characteristics.

The last property implies that for $\kappa_1, \kappa_2, \kappa_3$ in \mathcal{T} , the theta characteristic $\kappa_1 \otimes \kappa_2 \otimes \kappa_3^{-1}$ is even: in classical terms, \mathcal{T} is *syzygetic*. The existence of these vanishing thetanulls appears already in [F].

Proof : The first assertion follows from the previous lemma. Let κ be a σ -invariant theta characteristic; we have $\kappa = \pi^*L$ for some line bundle L on C with $\pi^*L^2 = K_C = \pi^*K_B$, which implies either $L^2 = K_B \otimes \rho$ or $L^2 = K_B$. In the first case we have

$$h^0(\kappa) = h^0(L) + h^0(L \otimes \rho) = h^0(L) + h^0(K_B \otimes L^{-1}) \equiv 0 \pmod{2}.$$

Since $\pi^*L \cong \pi^*(L \otimes \rho)$, we get 2^{2b-1} even theta characteristics of C .

In the second case L is a theta characteristic on B . We recall briefly the theory of theta characteristics on a curve, as explained for instance in [M1]. The group $V = \text{Pic}(B)[2]$ is a vector space over $\mathbb{Z}/2$, equipped with a symplectic form e , the *Weil pairing*. A quadratic form on V associated to e is a function $q : V \rightarrow \mathbb{Z}/2$ satisfying

$$q(\alpha + \beta) = q(\alpha) + q(\beta) + e(\alpha, \beta).$$

The set \mathcal{Q} of such forms is an affine space over V . Now the set of theta characteristics on B is also an affine space over V , which is in fact canonically isomorphic to \mathcal{Q} : the isomorphism associates to a theta characteristic L the form $q_L \in \mathcal{Q}$ defined by $q_L(\alpha) = h^0(L \otimes \alpha) + h^0(L) \pmod{2}$. Moreover the parity of L is equal to the Arf invariant $\text{Arf}(q_L)$.

Coming back to our situation, let L be a theta characteristic on B , and $\kappa = \pi^*L$; we have

$$h^0(\kappa) = h^0(L) + h^0(L \otimes \rho) \equiv q_L(\rho) \pmod{2}.$$

The function $q \mapsto q(\rho)$ is an affine function on \mathcal{Q} , hence it takes equally often the values 0 and 1. Taking into account the isomorphism $\pi^*L \cong \pi^*(L \otimes \rho)$, we get 2^{2b-2} even theta characteristics on C and 2^{2b-2} odd ones; summing up we obtain a).

Suppose $\kappa = \pi^*L$ is even, that is, $h^0(L) \equiv h^0(L \otimes \rho) \pmod{2}$; if we want $h^0(\kappa) > 0$, a good way (actually the only one if B is generic) is to choose L odd, that is, $\text{Arf}(q_L) = 1$. Equivalently, we look for forms $q \in \mathcal{Q}$ with $q(\rho) = 0$ and $\text{Arf}(q) = 1$.

Let ρ' be an element of V with $e(\rho, \rho') = 1$. ρ and ρ' span a plane $P \subset V$, such that $V = P \oplus P^\perp$. A form $q \in \mathcal{Q}$ is determined by its restriction to P and P^\perp , and we have $\text{Arf}(q) = \text{Arf}(q|_P) + \text{Arf}(q|_{P^\perp})$. The condition $q(\rho) = 0$ implies $\text{Arf}(q|_P) = q(\rho)q'(\rho) = 0$; so q is determined by $q(\rho') \in \mathbb{Z}/2$ and a form q' on P^\perp with Arf invariant 1. Since $\dim P^\perp = 2(b-1)$, there are $2^{b-2}(2^{b-1} - 1)$ such forms,

hence $2^{b-1}(2^{b-1} - 1)$ forms $q \in \mathcal{Q}$ with $q(\rho) = 0$ and $\text{Arf}(q) = 1$. Taking again into account the isomorphism $\pi^*L \cong \pi^*(L \otimes \rho)$, we obtain $2^{b-2}(2^{b-1} - 1) = 2^{g-2} - 2^{\frac{g-3}{2}}$ vanishing thetanulls on C .

They are contained in the affine space of theta characteristics $\kappa = \pi^*L$ with $q_L(\rho) = 0$, which has dimension $2b - 2 = g - 1$ and consists of even theta characteristics. ■

5. LOW GENUS

Let C be a non hyperelliptic curve of genus g . How many vanishing thetanulls can C have? The answer is well-known up to genus 5. There is no vanishing thetanull in genus 3, and at most one in genus 4 (which occurs if and only if the unique quadric containing the canonical curve is singular).

Suppose $g = 5$. If C is trigonal it admits at most one vanishing thetanull. Otherwise the canonical curve $C \subset \mathbb{P}^4$ is the base locus of a net Π of quadrics. The discriminant curve (locus of the quadrics in Π of rank ≤ 4) is a plane quintic with only ordinary nodes; these nodes correspond to the rank 3 quadrics of Π , that is to the vanishing thetanulls of C . Therefore C can have any number ≤ 10 of vanishing thetanulls; they are syzygetic [A]. The maximum 10 is attained by the so-called Humbert curves, for which all the quadrics in Π can be simultaneously diagonalized. They have an action of the group $(\mathbb{Z}/2)^4$, generated by 5 involutions with elliptic quotient.

Starting with $g = 6$ very little seems to be known. By Proposition 3.b), if C is bielliptic (that is, C admits an involution with elliptic quotient), it has 40 vanishing thetanulls. This can be slightly improved as follows. We take an elliptic curve B , a line bundle α of degree 2 on B , a point $p \in B$, and disjoint divisors A in $|\alpha(p)|$, A_1, A_2, A_3 in $|\alpha|$ which do not contain p . We put $\rho = \alpha^2(p)$ and $\bar{R} = A_1 + A_2 + A_3 + A + p$, and construct the double covering $\pi : C \rightarrow B$ associated to (ρ, \bar{R}) . The curve C has three extra vanishing thetanulls, namely $\mathcal{O}_C(\tilde{A}_i + \tilde{A}_j + \tilde{p})$ for $i < j$, where \tilde{A}_i and \tilde{p} are the lifts of A_i and p to C . Thus we get a genus 6 curve with 43 vanishing thetanulls; it is likely that one can do better.

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